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NONSINGULAR SMOOTHING REPRESENTATIONS FOR DIFFERENTIAL FLOWS ON MANIFOLDS

For general noncompact manifold and essentially nonlinear diffusion we derive the nonsingular raise of smoothness representations for semigroup derivatives which generalize results of [3]

1. Introduction.

Let us consider the parabolic Cauchy problem in $\mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = Hu(t,x) \\ u(0,x) = f(x) \in B \end{cases} \tag{1}$$

with operator

$$H = -\Delta + < F(x), \nabla \cdot >$$

where function $F \in C^\infty(\mathbb{R}^d)$ is monotone and grow at infinity with all its derivatives $F^{(i)}$ like:

$$\exists æ \quad \forall i \geq 1 \quad \|F^{(i)}(x) - F^{(y)}\| \leq C_i \|x-y\|(1+\|x\|+\|y\|)^{æ} \tag{2}$$

By standard theory (see, for example, [11]), the semigroup $P_t = e^{-tH}$ with generator H is associated with problem (1) and gives its solution $u(x,t) = P_t f(x)$. It is well-known that, if H is m - dissipative, densely defined operator in some Banach space B (say $C_b^n(\mathbb{R}^d)$), then semigroup P_t is strongly continuous in time and this space B is preserved under the action of semigroup, i.e. for initial function $f \in B$ the solution of (1) $u(t,\cdot) \in B$, for $t \in [0,T]$. But due to condition (2) the coefficients of operator H rapidly grow at infinity, thus the semigroup P_t is not strongly continuous in the space of bounded continuous functions $C_b^n(M)$. In this case analytical techniques do not give a grounds to conclude about the preservance of this space, and similar about the raise of smoothness in this scale: $P_t : C^n(\mathbb{R}^d) \mapsto C^{n+1}(\mathbb{R}^d)$.

In this paper we consider the problem (1), in the not strongly continuous case, when operator

$$H = \frac{1}{2} \sum_\alpha (< A_\alpha, \nabla >)^2 + < A_0, \nabla > \tag{3}$$

is considered on the non compact Riemannian manifold and has unbounded growing coefficients, generated by smooth vector fields $\{A_\alpha\}_{\alpha=1}^d$, A_0 on M, $dim M = d$.

Using the connection of semigroup theory with stochastic differential equations (SDE), given by formula

$$(P_t f)(x) = \mathbb{E} f(\xi_t^x)$$

with ξ_t^x to be the solution to special SDE on manifold M, we construct the representation for covariant derivatives of semigroup $\nabla_{k_n}^x ... \nabla_{k_1}^x P_t f(x)$ in terms of pure (without derivatives) function f, of the type:

$$\nabla_{k_n}^x ... \nabla_{k_1}^x P_t f(x) = \mathbb{E} f(\xi_t^x) \Psi_{k_1 ... k_n} \tag{4}$$

with some kernels $\Psi_{k_1...k_n}$. Good estimates on these kernels will imply the raise of smoothness property for semigroup P_t.

Similar representation for the first order derivative of P_t was obtained in [7, 9] in the case of compact Riemannian manifold.

2. SDE on manifolds: notations, assumptions and auxiliary results.

Let M be smooth connected d-dimensional Riemannian manifold with metric tensor g_{ij}. Using that operator H is second order differential operator semigroup P_t can be reconstructed by the the solution of stochastic differential equation on M:

$$\begin{cases} \delta\xi_t^x = A_0(\xi_\tau^x)d\tau + \sum\limits_\alpha A_\alpha(\xi_\tau^x)\delta W_\tau^\alpha \\ \xi_0^x = x \in M \end{cases} \tag{5}$$

by formula

$$(P_t f)(x) = \mathbb{E}\, f(\xi_t^x). \tag{6}$$

Here $\mathbb{E}$ denotes mathematical expectation with respect to the canonically realized Wiener measure $\mathcal{W}$ on the probability space $\Omega = C_0(I\!R_+, I\!R^d)$, W_τ^α, $\alpha = 1, ..., d$ denote d independent $I\!R^1$-valued Wiener processes, δW_τ is Stratonovich differential.

Correct definition of equation (6) on the manifold is an important question, which was widely discussed in the literature. There are several approaches to the solution of this problem (see, for example, [4, 8, 10, 12]). In this paper under the solution of equation (5) we understand the adapted process ξ_t^x, such that for any infinitely differentiable function f the following stochastic differential equation is true in $I\!R^1$:

$$f(\xi_t^x) = f(x) + \int_0^t (A_0 f)(\xi_\tau^x)d\tau + \sum_\alpha \int_0^t (A_\alpha f)(\xi_\tau^x)\delta W_\tau^\alpha$$

Choosing $f(x) = x^k$ we obtain the equation in local coordinates x^k in some vicinity U_x.

Let us impose on the coefficients of operator H and geometry of manifold the following conditions.

A1) Dissipativity. For some fixed point $o \in M$ and any constant $C \in I\!R^1$ there exists such constant $K_C \in I\!R^1$ that $\forall\, x \in M$

$$< \widetilde{A_0}(x), \nabla^x \rho^2(x, o) > + C\sum_{\alpha=1}^d \|A_\alpha(x)\|^2 \le K_C(1 + \rho^2(x, o))$$

Here

$$\widetilde{A_0} = A_0 + \frac{1}{2}\sum_\alpha \nabla_{A_\alpha} A_\alpha,$$

where ∇_{A_α} denotes covariant derivative in direction of vector field A_α, ∇^x means that covariant derivative ∇ acts on function $\rho^2(x, o)$ on the first variable x, ρ is a metric on manifold M. Further the norms of vector field and pairing $< \cdot, \cdot >$ are taken in corresponding tangent spaces.

A2) Coercitivity: for any constants $C, C' \in I\!R^1$ there exists such constant $K_C \in I\!R^1$, that for any $x, y \in M$ and h:

$$< \nabla\widetilde{A_0}[h], h > + C\sum_{\alpha=1}^d \|\nabla A_\alpha[h]\|^2 + C'\sum_{\alpha=1}^d < R(A_\alpha, h)A_\alpha, h > \le K_C\|h\|^2$$

Here expression

$$< R(A_\alpha, h)A_\alpha, h >= g_{mn}R_{p\ \ell q}^{\ m}\ A_\alpha^p h^n A_\alpha^\ell h^q$$

is defined in terms of $(1,3)$ curvature tensor $R_{p\ \ell q}^{\ m}$ and metric tensor g_{mn}.

A3) Nonlinear growth of coefficients. For all $n \in I\!N$ there exists constants $æ_0, æ_\alpha, æ_R$ such that

$$\|\nabla_\gamma \widetilde{A_0}(x)\| \leq (1 + \rho(x, o))^{æ_0}$$
$$\|\nabla_\gamma A_\alpha(x)\| \leq (1 + \rho(x, o))^{æ_\alpha}$$
$$\|\nabla_\gamma R(x)\| \leq (1 + \rho(x, o))^{æ_R}$$

where $\nabla_\gamma = \nabla_{j_1} ... \nabla_{j_n}$, $\gamma = \{j_1, ..., j_n\}$.

A4) Non degenerate diffusion coefficient. There exists constant $æ_\varepsilon > 0$ such that

$$\inf_x \frac{\|A_\alpha(x)\|}{(1 + \rho^2(x, o))^{æ_\varepsilon}} \geq \varepsilon > 0.$$

For further use we need some notations and concepts from the Malliavin calculus.

Let $\mathcal{J}$ denote the set of all adapted continuous integrable $I\!R^d$-valued processes $u_t(\omega)$ such that $\mathbb{E} \int\limits_0^T |u_t|^p dt \leq \infty$ for all $T, p > 0$. Recall that random function $F(\omega)$, defined on the Wiener space $\omega \in \Omega = C_0(I\!R_+, I\!R^d)$, is *stochastically differentiable* in the direction $u \in \mathcal{J}$ if on the set of full measure there exists derivative in space $\bigcap\limits_{p \geq 1} L_p(\Omega, \mathcal{W})$:

$$D_u F(\omega) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(\{\omega^\sigma(t) + \varepsilon \int\limits_0^t u_s^\sigma ds\}_{t \in I\!R^+}^{\sigma=1,..,d}) \tag{7}$$

It is well-known (see [5, 15, 16, 17, 19]), that stochastic derivative D_u possesses the properties, similar to usual derivatives:

$$\begin{aligned} &1^o.\ \ D_u f(\xi^1, ..., \xi^n) = \sum_{i=1}^d \partial_i f(\xi^1, ..., \xi^n) D_u \xi^i \\ &2^o.\ \ D_u \int\limits_0^t f_\tau d\tau = \int\limits_0^t D_u f_\tau d\tau \\ &3^o.\ \ D_u \int\limits_0^t \sum_{\sigma=1}^d g_\sigma(\tau)\delta W_\tau^\sigma = \int\limits_0^t \sum_{\sigma=1}^d D_u g_\sigma(\tau)\delta W_\tau^\sigma + \int\limits_0^t \sum_{\sigma=1}^d u_\tau^\sigma g_\sigma(\tau) d\tau \end{aligned} \tag{8}$$

Moreover, it is true the following integration-by-parts formula [17, 20], which means that stochastic derivative is adjoint operation to the integration with respect to the Wiener measure:

$$\mathbb{E} D_u F = \mathbb{E} F \int\limits_0^\infty \sum_{\sigma=1}^d u_\tau^\sigma dW_\tau^\sigma \tag{9}$$

which implies, in particular, that

$$\mathbb{E}(D_u F)G = -\mathbb{E} F D_u G + \mathbb{E} F\, G \int_0^\infty \sum_{\sigma=1}^d u_\tau^\sigma dW_\tau^\sigma \tag{10}$$

To see the structure of equation on the covariant derivative we write

$$\delta(\frac{\partial \xi^m}{\partial x^k}) = \frac{\partial A_0^m(\xi)}{\partial x^k}dt + \frac{\partial A_\sigma^m(\xi)}{\partial x^k}\delta W^\sigma$$

Adding and subtracting terms with connection we collect the covariant derivatives of vector fields $A_\alpha(\xi_t^x)$, $A_0(\xi_t^x)$ and apply the symmetry of connection on lower indexes.

$$\delta(\frac{\partial \xi^m}{\partial x^k}) = \Big(\nabla_j A_0^m(\xi)\frac{\partial \xi^j}{\partial x^k} - \Gamma_{p\ q}^{\ m}(\xi)\frac{\partial \xi^p}{\partial x^k}A_0^q(\xi)\Big)dt + \Big(\nabla_j^x A_\sigma^m(\xi)\frac{\partial \xi^j}{\partial x^k} - \Gamma_{p\ q}^{\ m}(\xi)\frac{\partial \xi^p}{\partial x^k}A_\sigma^q(\xi)\Big)\delta W^\sigma$$

Collecting the differential $\delta\xi = A_0(\xi)dt + A_\sigma(\xi)\delta W^\sigma$ we obtain equation on *ordinary derivative* of process ξ with respect to the initial data

$$\frac{\partial \xi_t^m}{\partial x^k} = \int_0^t \Gamma_{p\ q}^{\ m}(\xi)\frac{\partial \xi^p}{\partial x^k}\delta\xi^q + \int_0^t \nabla_j A_0^m(\xi)\frac{\partial \xi^j}{\partial x^k}ds + \int_0^t \nabla_j A_\sigma^m(\xi)\frac{\partial \xi^j}{\partial x^k}\delta W^\sigma \quad (11)$$

Using the properties of stochastic derivatives (8), in particular that

$$D_u A^m(\xi_t) = \partial_j A^m(\xi_t) D_u \xi_t^j \quad (12)$$

we write

$$\delta(D_u\xi_t^m) = \Big(\partial_j A_0^m(\xi)D_u\xi_t^j + A_\sigma^m(\xi)u^\sigma\Big)dt + \partial_j A_\sigma^m(\xi)D_u\xi_t^j\delta W^\sigma \quad (13)$$

Adding and subtracting terms with connection we form the covariant derivatives:

$$\delta(D_u\xi_t^m) = \Big(\nabla_j A_0^m(\xi)D_u\xi^j - \Gamma_{p\ q}^{\ m}(\xi)D_u\xi^p A_0^q(\xi) + A_\sigma^m(\xi)\,u^\sigma\Big)dt + \\ +\Big(\nabla_j A_\sigma^m(\xi)D_u\xi^j - \Gamma_{p\ q}^{\ m}(\xi)D_u\xi^p A_\sigma^q(\xi)\Big)\delta W^\sigma$$

Finally, separating the differential $\delta\xi$ we achieve the equation on *stochastic derivative* of process ξ

$$D_u\xi_t^m = \int_0^t A_\sigma^m(\xi)u^\sigma ds - \int_0^t \Gamma_{p\ q}^{\ m}(\xi)D_u\xi^p\delta\xi^q + \int_0^t \nabla_j A_0^m(\xi)D_u\xi^j ds + \int_0^t \nabla_j A_\sigma^m(\xi)D_u\xi^j\delta W^\sigma \quad (14)$$

3. Invariant representation for semigroup derivatives.

This section we would like to begin with some foregoing discussions. Let us apply the integration by parts formula (9) to functional $F = f(\xi)\Psi$, then we have

$$\mathbb{E}D_u\Big[f(\xi_t)\Psi\Big] = \mathbb{E}f(\xi_t)\Psi\int_0^t\sum_{\sigma=1}^d u_\tau^\sigma dW_\tau^\sigma$$

Properties of stochastic derivative D_u (8) imply:

$$\mathbb{E}f'(\xi_t)D_u\xi_t\,\Psi = \mathbb{E}f(\xi_t)\Big\{\Psi\int_0^t\sum_{\sigma=1}^d u_\tau^\sigma dW_\tau^\sigma - D_u\Psi\Big\}$$

or, redenoting $D_u\xi_t\,\Psi$ by Φ, we have

$$\mathbb{E}\,f'(\xi_t)\Phi = \mathbb{E}\,f(\xi_t)\Big\{\frac{\Phi}{D_u\xi_t}\int\limits_0^t\sum_{\sigma=1}^d u_\tau^\sigma dW_\tau^\sigma - D_u\big(\frac{\Phi}{D_u\xi_t}\big)\Big\}$$

Using that $\dfrac{\partial}{\partial x^k}P_t f(x) = \mathbb{E}\,\partial_j f(\xi)\dfrac{\partial\xi^j}{\partial x^k}$ we conclude

$$\frac{\partial}{\partial x^k}P_t f(x) = \mathbb{E}\,f(\xi_t)\Big\{\frac{\partial_k\xi_t}{D_u\xi_t}\int\limits_0^t\sum_{\sigma=1}^d u_\tau^\sigma dW_\tau^\sigma - D_u\Big(\frac{\partial_k\xi_t}{D_u\xi_t}\Big)\Big\} \tag{15}$$

Actually, formula (15) shows that to obtain the nonsingular representation of semigroup derivative $\dfrac{\partial}{\partial x}P_t f(x)$ it is necessary to guarantee, that expression in brackets $\{...\}$ is not singular. The simplest way to do this is to express $D_u\xi_t$ as a function of $\partial_k\xi$. This idea lies in the grounds of our investigation.

THEOREM 1. *Let conditions A1) – A4) be fulfilled. Then there exists stochastic direction* $z_k \in \mathcal{J}$ *such that*

$$D_{z_k}\xi_t^m = t\frac{\partial\xi_t^m}{\partial x^k} \tag{16}$$

and the following estimate is true:

$$\mathbb{E}\Big(\int_0^t z_k^\sigma dW^\sigma\Big)^{2q} \le K\,t^q e^{M\,t}(1+\rho^2(x,o))^{2q æ_1} \tag{17}$$

Proof. 1. Let us take direction $u = z_k$ chosen so that

$$\frac{\partial\xi_t^m}{\partial x^k} = A_\sigma^m(\xi_t)z_k^\sigma \tag{18}$$

Substituting in (14) instead of $D_u\xi_t$ its value in the direction $u = z_k$: $D_{z_k}\xi^m = t\dfrac{\partial\xi^m}{\partial x^k}$, using the equation (11), and relation (12), it is not difficult to check that equation (14) is valid identically. In [2] it was proved the solvability and moment estimates on the solutions of coercitive equations (11) and (14). From the uniqueness of solutions of these equations it follows result (16).

2. The Doob inequality [6] gives:

$$\mathbb{E}\Big(\int_0^t z_s^\sigma\,dW_s^\sigma\Big)^{2q} \le K_q t^{q-1}\mathbb{E}\int_0^t \|z_s\|^{2q}\,ds$$

In [1] it was proved the following nonlinear quasi-contractive estimate

$$r_n(\xi^x, t) \le e^{Kt}r_n(x,0) \tag{19}$$

for nonlinear expression

$$r_n(\xi,t) = \sum_{j=1}^n \mathbb{E}\,p_j(\rho^2(\xi_t^x,0))\|(\nabla^x)^j\xi_t^x\|^{q/j}$$

where

$$\|(\mathbb{W}^x)^j \xi_t^x\|^2 = g_{mn}(\xi_t^x) \prod_{\sigma=1}^{j} g^{i_\sigma k_\sigma}(x) \mathbb{W}^x_{i_1,...,i_\sigma} \xi^m \mathbb{W}^x_{k_1,...,k_\sigma} \xi^n$$

in term of some generalization of covariant Riemannian derivative

$$\mathbb{W}_k \xi^m = \nabla_k \xi^m = \frac{\partial \xi^m}{\partial x^k}$$
$$\mathbb{W}_k(\mathbb{W}_\gamma \xi^m) = \nabla_k(\mathbb{W}_\gamma \xi^m) + \Gamma_{p\ q}^{\ m}(\xi)\mathbb{W}_\gamma \xi^p \nabla_k \xi^q$$

In particular, nonlinear estimate (19) implies:

$$\mathbb{E}(1+\rho^2(\xi_t, o))^{2q\ae} \|\nabla \xi_t\|^{2q} \leq M e^{N t}(1+\rho^2(x,o))^{2q\ae}$$

Therefore

$$\mathbb{E}(\int_0^t z_s^\sigma dW_s^\sigma)^{2q} \leq K_q t^{q-1} \mathbb{E}\int_0^t (1+\rho^2(\xi_s,o))^{2q\ae_1} \|\nabla \xi_s\|^{2q} ds \leq K t^q e^{N t}(1+\rho^2(x,o))^{2q\ae_1}$$

which implies the statement. □

Let us introduce the following notation:

$$Y_k = t\nabla_k - D_{z_k} + \int_0^t \sum_{\sigma=1}^{d} z_k^\sigma dW_s^\sigma \tag{20}$$

where z_k is a specially chosen direction (18).

THEOREM 2. *High order covariant derivative of semigroup P_t permits representation:*

$$\nabla^x_\gamma P_t f(x) = \frac{1}{t^{|\gamma|}} \mathbb{E} f(\xi_t^x) Y_\gamma 1 \tag{21}$$

Above $Y_\gamma = Y_{k_n} ... Y_{k_1}$ for a set $\gamma = \{k_1, ..., k_n\}$.

Proof. 1. Base of induction follows from the Kolmogorov representation for semigroup for $j = 0$.

2. Inductive step. Let (21) hold for some γ, $\gamma = \{k_1, ..., k_n\}$. We write its next $|\gamma| + 1$ - derivative, using the representation for derivative of order $|\gamma|$:

$$\nabla_k \nabla_\gamma P_t f(x) = \partial_k^x \nabla_\gamma P_t f(x) - \sum_{j\in\gamma} \Gamma_{j\ k}^{\ h}(x) \nabla_{\gamma|_{j=h}} P_t f(x) =$$

$$= \partial_k^x (\frac{1}{t^{|\gamma|}} \mathbb{E} f(\xi) Y_\gamma 1) - \frac{1}{t^{|\gamma|}} \sum_{j\in\gamma} \Gamma_{k\ j}^{\ h}(x) \mathbb{E} f(\xi) Y_{\gamma|_{j=h}} 1 =$$

$$= \frac{1}{t^{|\gamma|}} \mathbb{E} f(\xi) \{\partial_k^x Y_\gamma 1 - \sum_{j\in\gamma} \Gamma_{k\ j}^{\ h}(x) Y_{\gamma|_{j=h}} 1\} + \frac{1}{t^{|\gamma|}} \mathbb{E} \partial_k^x f(\xi) Y_\gamma 1 \tag{22}$$

Theorem 1implies:

$$\partial_k^x f(\xi^x) = \frac{\partial f}{\partial \xi^j} \frac{\partial \xi^j}{\partial x^k} = \partial_j f \frac{\partial \xi^j}{\partial x^k} = \frac{1}{t} \partial_j f D_{z_k} \xi^j = \frac{1}{t} D_{z_k} f(\xi) \tag{23}$$

Therefore

$$\nabla_k \nabla_\gamma P_t f(x) = \frac{1}{t^{|\gamma|+1}} \mathbb{E} f(\xi_t^x)\{t\nabla_k - D_{z_k} + \int_0^t \sum_{\sigma=1}^d z_k^\sigma dW^\sigma\} Y_\gamma 1.$$

□

4. Concluding remarks on orthoframe approach.

Let us remind that the representation (21) is obtained in the case, when the stochastic differential equation has the form (5), and therefore, generator is given by (3). It is well-known, that in the case of non zero Euler number, the manifold do not permit the existence of globally defined smooth unit vector fields. Therefore, it is impossible to find globally defined fields A_α, A_0 such that operator H becomes Laplacian.

For this purpose we may consider the system, written in local coordinates:

$$\begin{cases} \xi^j(t) = x^k + \int_0^t B_k^j(\xi) e_\alpha^k(t) dW^\alpha + \int_0^t A^j(\xi) dt \\ e_\beta^i(t) = e_\beta^i(0) - \int_0^t \Gamma_{m\ k}^{\ i}(\xi) e_\beta^k(t) \delta\xi^m \end{cases} \tag{24}$$

where W^α, $\alpha = 1, ..., d$ denote d $\mathbb{R}^1$-valued independent Wiener processes, $B \in T_1^1(M)$ and A should be specified further to obey the Ito transformation law.

Remark that in system (24) the equation on ξ_t on base space is related with the evolution of orthoframe field $\vec{e}_\alpha = (e_\alpha^i)_{i=1}^d$, $\alpha = 1, ..., d$, that form a basis in $T_x M$. Thus $g_{ij} e_\alpha^i e_\beta^j = \delta_{\alpha\beta}$ or in matrix notations

$$\mathcal{E}^* G \mathcal{E} = Id \tag{25}$$

which, in particular, implies

$$\sum_{\alpha=1}^d e_\alpha^i e_\alpha^j = g^{ij}$$

where g^{ij} are elements of inverse metric tensor $G^{-1} = (g^{ij})$.

Therefore, the application of Ito formula leads to the generator of process ξ_t (24)

$$H = \frac{1}{2} Tr\Big(B\mathcal{E}(B\mathcal{E})^* \partial^2\Big) + A\partial = \frac{1}{2} Tr\Big(BG^{-1}B^* \partial^2\Big) + A\partial$$

that does not depend on the choice of particular orthoframe field $\vec{e}_\alpha$.

In particular, for $B = Id$ one gets the Laplace - Beltrami operator

$$\Delta = \frac{1}{2} \frac{1}{\sqrt{detG}} \frac{\partial}{\partial x^i} \Big(g^{ij} \sqrt{detG} \frac{\partial}{\partial x^j}\Big) =$$

$$= \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \Big(\ln(detG)\Big) \frac{\partial}{\partial x^j}$$

if $A = \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \Big(\ln(detG)\Big) \frac{\partial}{\partial x^j}$. In this case process $(\xi_t, \vec{e}_\alpha(t))$ is known as Ito development and has important applications to the analysis on Wiener space [5, 14]. We also remark that the general construction of general diffusions on manifold, without introduction of orthofield $\vec{e}$ were developed by different authors, see [4, 7, 13, 18] and references therein.

By the scheme of previous section, it is clear that the main point in the seeking of nonsingular semigroup representations is to find such direction $u \in \mathcal{J}$ that

$$D_{u(v)}\xi_t = t\partial_v \xi_t$$

for some smooth vector field $v \in TM$.

To proceed further we need the generalization of stochastic derivative, introduced in previous section, which is known as *Bismut derivative* $D_{(q,h)}$ for $so(d) \times I\!R^d$-valued process (q, h):

$$D_{(q,h)}F(\omega) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(\int_0^t \exp(\varepsilon q_s) dW_s + \varepsilon h_t)$$

Derivative $D_{(q,h)}$ is well defined on cylindric functions and by standard procedure permit the continuation on the functionals on Wiener space [5, 19]. As a consequence of Girsanov theorem the following integration by parts formula can be proved (see, for example [20]):

$$\mathbb{E}D_{(q,h)}F = \mathbb{E}F \int_0^t \dot{h}(\omega, t) dW_t \tag{26}$$

For Bismut derivative D_u it is not difficult to prove that for processes X, Y with differentials

$$dX = X^1 dW + X^2 dt \qquad dY = Y^1 dW + Y^2 dt$$

and $u = (q, h)$ with skew-symmetric $q \in so(d)$

$$D_u \int X_t \delta Y_t = \int (D_u X_t)\delta Y_t + \int X_t \delta(D_u Y_t) \tag{27}$$

THEOREM 3. *Let process (ξ, e) be governed by equation (24). Then for process $u(v) = (q(v), h(v))$ given by*

$$\begin{aligned} q_\alpha^\beta(t) &= \int_0^t \hat{e}_i^\beta [\partial_v e_\alpha^i + \Gamma_m{}^i{}_k e_\alpha^k (\partial_v \xi^m)] ds \\ \dot{h}_t^m &= \hat{e} B^{-1}(\xi) \partial_v \xi_t^m \end{aligned} \tag{28}$$

relation holds

$$D_{u(v)}\xi_t^x = t\partial_v \xi_t^x$$

Proof. Let us write for process ξ_t^x given by (24) the equation on stochastic derivative $D_u \xi_t^x$ and $t\partial_v \xi_t^x$, where $\partial_v \xi_t^x$ be ordinary derivative of process ξ with respect to the initial data x in the direction of vector field $v \in TM$:

$$t\partial_v \xi_t^k = \int_0^t \partial_v \xi_s^k ds + \int_0^t s(\partial_n B_m^k)(\partial_v \xi_s^n) e_\alpha^m dW_s^\alpha + \int_0^t s B_m^k \partial_v e_\alpha^m dW_s^\alpha + \int_0^t s(\partial_n A^k)\partial_v \xi_s^n ds \tag{29}$$

$$\begin{aligned} D_{u(v)}\xi_t^k &= \textstyle\int_0^t (\partial_n B_m^k)(D_{u(v)}\xi_s^n) e_\alpha^m(s) dW_s^\alpha + \int_0^t B_m^k \Big(D_{u(v)} e_\alpha^m(s)\Big) dW_s^\alpha + \\ &+ \textstyle\int_0^t (\partial_n A^k)(D_{u(v)}\xi_s^n) ds + \int_0^t B_m^k e_\alpha^m \Big(\dot{h}_s^\alpha(v) ds + q_\beta^\alpha(v) dW_s^\beta\Big) \end{aligned} \tag{30}$$

Imposing condition $D_{u(v)}\xi^x_t = t\partial_v \xi^x_t$ we have that integrals with $\partial_n B^k_m$ and $\partial_n A^k$ disappear. Making two differentials equal we have

$$\text{at } dW \qquad D_{u(v)}e^m_\alpha + e^m_\gamma q^\gamma_\alpha(v) = t\partial^x_v e^m_\alpha$$
$$\text{at } dt \qquad \partial^x_v \xi^k_t = B^k_m(\xi) e^m_\alpha h^\alpha_t$$

For process $e^i_\beta(t)$ governed by (24) we have:

$$t\partial_v e^i_\alpha = \int_0^t (\partial_v e^i_\alpha) ds - \int_0^t s\partial_v \left(\Gamma^{\ i}_{m\ k} e^k_\alpha\right)\delta\xi^m)_s - \int_0^t s\Gamma^{\ i}_{m\ k} e^k_\alpha \delta(\partial_v \xi^m_s)$$

$$D_{u(v)} e^i_\alpha = -\int_0^t D_{u(v)}\left(\Gamma^{\ i}_{m\ k} e^k_\alpha\right)\delta\xi^m_s - \int_0^t \Gamma^{\ i}_{m\ k} e^k_\alpha \delta\left(D_{u(v)}\xi^m_s\right)$$

Using that $D_{u(v)}\xi^x_t = t\partial_v \xi^x_t$ implies $t\partial_v f(\xi) = D_u f(\xi)$, in particular for $f(\xi) = \Gamma^{\ i}_{m\ k}(\xi)$, and properties of Stratonovich integral, we have

$$e^i_\gamma q^\gamma_\alpha = t\partial_v e^i_\alpha - D_{u(v)} e^i_\alpha = \int_0^t (\partial_v e^i_\alpha) ds - \int_0^t \Gamma^{\ i}_{m\ k}\left(s\partial_v e^k_\alpha - D_{u(v)} e^k_\alpha\right)\delta\xi^m_s -$$

$$-\int_0^t s\Gamma^{\ i}_{m\ k} e^k_\alpha \delta(\partial_v \xi^m_s) + \int_0^t \Gamma^{\ i}_{m\ k} e^k_\alpha \delta(s\partial_v \xi^m) =$$

$$= \int_0^t (\partial_v e^i_\alpha) ds + \int_0^t \Gamma^{\ i}_{m\ k} e^k_\alpha (\partial_v \xi^m_s) ds - \int_0^t \Gamma^{\ i}_{m\ k}[e^k_\gamma q^\gamma_\alpha]\delta\xi^m_s$$

Let us denote by $\hat{e}^\beta_i$ the elements of matrix $\hat{\mathcal{E}}$ inverse to $\mathcal{E}$, then $q^\beta_\alpha = \hat{e}^\beta_i (e^i_\gamma q^\gamma_\alpha)$. Accounting that $q(0) = 0$ we have

$$q^\beta_\alpha(t) = \int_0^t (e^i_\gamma q^\gamma_\alpha)\delta\hat{e}^\beta_i + \int_0^t \hat{e}^\beta_i \delta(e^i_\gamma q^\gamma_\alpha) =$$

$$= \int_0^t e^i_\gamma q^\gamma_\alpha \Gamma^{\ j}_{m\ i} \hat{e}^\beta_j \delta\xi^m_s + \int_0^t \hat{e}^\beta_i \left(\partial_v e^i_\alpha + \Gamma^{\ i}_{m\ k} e^k_\alpha (\partial_v \xi^m)\right) ds - \int_0^t \hat{e}^\beta_i \Gamma^{\ i}_{m\ k} e^k_\gamma q^\gamma_\alpha \delta\xi^m_s$$

Therefore

$$q^\beta_\alpha(t) = \int_0^t \hat{e}^\beta_i \left(\partial_v e^i_\alpha + \Gamma^{\ i}_{m\ k} e^k_\alpha (\partial_v \xi^m)\right) ds \qquad (31)$$

□

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